Plastic Stability of Spherical Plates

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The plastic stability of spherical plate elements is treated, and plasticity reduction factors are obtained for axially loaded circular flat plates and spherical plates under lateral pressure in the transition region. Linear stability theory is used to investigate the effect of various boundary conditions and also to determine in some detail the regions in which the axisymmetric and asymmetric buckling modes govern. These results, obtained first for the elastic stability case, are then used to simplify the approach to the plastic stability case.

Nomenclature

A_{ij} = plasticity coefficient matrix
A = plasticity parameter = $(\frac{1}{2})A_{12}/A_{11}$
B = axial rigidity: elastic = $Et/(1 - \nu_e^2)$; plastic = $E_s t/(1 - \nu_p^2)$
D = flexural rigidity: elastic = $Et^3/12(1 - \nu_e^2)$; plastic = $E_s t^3/12(1 - \nu_p^2)$
d = diameter of the base circle (dimensionless)
E_s = secant modulus
E_t = tangent modulus
E = elastic modulus
F = stress function
k = buckling coefficient = $\sigma t d^2/\pi^2 D$
m = number of nodal circles
M = bending moment resultant per unit length
n = number of nodal lines
N = direct stress resultant per unit length
p = external pressure
r = radial coordinate (dimensionless)
R = radius of the sphere of which the cap is a part (dimensionless)
t = thickness of the shell (dimensionless)
u_ϕ, u_θ, w = dimensionless displacements
Z = dimensionless shell curvature parameter = $(d^2/Rt)(1 - \nu_p^2)^{1/2}$
α = dimensionless buckling stress: elastic = $(\sigma t/D)^{1/2}$; plastic = $(\sigma t/DA_{11})^{1/2}$
β = dimensionless shell geometry parameter: elastic = $(Et/DR^2)^{1/2}$; plastic = $[E_t t/DA_{11}{}^2R^2]^{1/2}$
ϵ = direct strain variation
$\bar{\eta}$ = plasticity reduction factor
ν_e = elastic Poisson ratio
ν_p = plastic Poisson ratio = $\frac{1}{2}$
σ = constant compressive stress at buckling = $pR/2t$
χ = curvature variation
∇^2 = Laplacian operator, $\partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2) \times \partial^2/\partial\theta^2$

Introduction

THE use of spherical plates as pressure vessel closures in aerospace vehicles has generated continued interest in the stability of spherical plates under external pressure. The term spherical plates refers to those spherical elements that fall in the transition range between axially loaded circular flat plates where boundary conditions are significant and spherical shells where boundary conditions tend to be of less significance.

The basic objective of this paper is to investigate the plastic stability of spherical plates. New results in terms of plasticity reduction factors are thus obtained for flat circular plates under edge compressive loads and spherical plates under lateral pressure in the transition region. For the limiting case of the full sphere, results are obtained in the same form as given in Ref. 1.

As an additional part of this investigation, the regions in which the axisymmetric and asymmetric buckling modes govern are examined in some detail. This is accomplished by examining the elastic stability of spherical plates for various boundary conditions first and then proceeding to the plastic stability case utilizing the conclusions drawn from the elastic stability study.

In the derivation of the governing equations for the spherical plates considered herein, certain simplifying assumptions are introduced. These include the use of linear stability theory, deformation type plasticity laws and boundary conditions that permit the ready solution of the governing differential equations but which are not readily interpreted in terms of physical conditions. In addition, the influence of prebuckling deformations are neglected.

In view of these assumptions and the fact that linear elastic stability theory is known to result in buckling loads higher than experimental results for spherical plates and shells, the significance of the results obtained herein for the plasticity reduction factors bear some discussion in terms of design applications. Although there is considerable literature on the nonlinear elastic buckling of clamped spherical plates, the current status can be stated rather briefly in terms of Huang's theory[2] and Krenzke and Kiernan's experiments.[3] The theory that accounts for axisymmetric prebuckling deformations followed by asymmetric buckling beyond a certain value of curvature parameter is in substantial agreement with the carefully obtained experimental results of Ref. 3. As a consequence, the plasticity reduction factors obtained herein should be used in conjunction with the best available elastic stability results for design purposes.

Governing Equations

When a spherical shell buckles under external pressure, the buckle wavelength is confined to a small portion of the surface, and the critical stress can be evaluated without specific reference to the edge boundary conditions. As another limiting case, let us consider the instability of a flat circular plate under axial compressive loading. Here the half wavelength of the buckling mode is of the same order of magnitude as the plate diameter and the buckling stress is considerably influenced by the edge support.

It is evident from these two examples that the transition from the flat-plate case to that of the full sphere can be effected by considering a series of spherical plates defined in the usual manner of shallow shell theory in which the vertical rise is small compared to a characteristic horizontal length; the buckle half-wavelength in the case of the full sphere and the spherical plate or the diameter of base circle in the case of the flat plate. The spherical plate solution will readily

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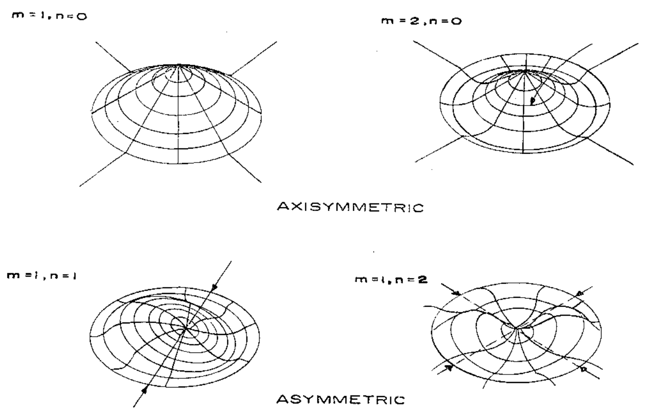

Fig. 1 Shapes of a few of the normal modes of buckling of a clamped circular plate.

yield the results of the circular-plate case by letting the appropriate term containing the height of the shell vanish. It is also possible to obtain the solution of the full sphere as a singular case of the same equations.

In deriving the governing equations for the stability of spherical plates, the assumptions of shallow shell theory lead to simplifications in both the equilibrium and strain-displacement relationships. In the following, the distance from the axis of the shell to the edge of the cap is taken as the unit of length, and hence all lengths are essentially dimensionless.

The strain displacement relationships are:

$$\epsilon_\theta = u_\phi/r + (1/r)(\partial u_\theta/\partial\theta) + w/R$$
$$\chi_\theta = (1/r)(\partial w/\partial r) + (1/r^2)(\partial^2 w/\partial\theta^2) \qquad (1)$$

$$\epsilon_\phi = \partial u_\phi/\partial r + w/R \qquad \chi_\phi = \partial^2 w/\partial r^2 \qquad (2)$$

$$\epsilon_{\theta\phi} = \tfrac{1}{2}[(1/r)(\partial u_\phi/\partial\theta) + r(\partial/\partial r)(u_\theta/r)]$$
$$\chi_{\theta\phi} = (\partial/\partial r)(1/r\,\partial w/\partial\theta) \qquad (3)$$

In the preceding equations, ϵ, χ denote the direct strain and curvature variations; u_ϕ, u_θ, and w the displacements in the meridional, circumferential, and normal directions, respectively. The direct strain variations ϵ_ϕ, ϵ_θ, $\epsilon_{\theta\phi}$, are seen to satisfy a compatibility relationship:

$$(1/r^2)\partial^2\epsilon_\phi/\partial\theta^2 - (1/r)\partial\epsilon_\phi/\partial r + (2/r)\partial\epsilon_\theta/\partial r +$$
$$\partial^2\epsilon_\theta/\partial r^2 - (1/r^2)(\partial^2/\partial r\partial\theta)(r^2\epsilon_{\theta\phi}) = 1/r\nabla^2 w \qquad (4)$$

The equilibrium equations for the buckling problem of a shallow spherical cap under external pressure, consistent with the strain-displacement relations [Eqs. (1–3)], are

$$\partial(rN_\phi)/\partial r + \partial N_{\phi\theta}/\partial\theta - N_\theta = 0 \qquad (5)$$

$$\partial(rN_{\phi\theta})/\partial r + \partial N_\theta/\partial\theta + N_{\phi\theta} = 0 \qquad (6)$$

$$(1/r)[\partial^2(rM_\phi)/\partial r^2 + 2\partial^2(M_{\theta\phi})/\partial r\partial\theta - \partial M_\theta/\partial r +$$
$$(2/r)\partial M_{\theta\phi}/\partial\theta] + (1/r^2)\partial^2 M_{\theta\phi}/\partial\theta^2 +$$
$$1/R(N_\theta + N_\phi) + p + \sigma t\,\nabla^2 w = 0 \qquad (7)$$

In Eqs. (5–7), N, M refer to the direct stress and moment resultants, respectively; p is the external pressure, σ the constant compressive stress ($= pR/2t$) at buckling, R the radius of the sphere of which the cap is a part, t the thickness of the shell, and ∇^2 the Laplacian operator.

From Eqs. (5) and (6) it is readily seen that a stress function F can be introduced such that they are satisfied identically. The direct stress resultants then are derivable from the stress function F as shown:

$$N_\phi = (1/r)\partial F/\partial r + (1/r^2)\partial^2 F/\partial\theta^2$$
$$N_\theta = \partial^2 F/\partial r^2 \qquad (8)$$
$$N_{\phi\theta} = -(\partial/\partial r)[(1/r)\partial F/\partial\theta]$$

The plastic stability theory used herein, following Ref. 1, is based on a deformation theory of plasticity. Hence the stress-strain relationships for the spherical-plate case are similar in form in both elastic and plastic ranges. It would be advantageous to derive the governing equations in terms of the plastic coefficients so that the elastic results are readily obtained by modifying the coefficients suitably.

The stress resultant-strain relationships in the plastic range[1] for the spherical plate are

$$N_\phi = BA_{11}(\epsilon_\phi + \bar{A}\epsilon_\theta) \qquad M_\phi = DA_{11}(\chi_\phi + \bar{A}\chi_\theta) \qquad (9)$$

$$N_\theta = BA_{11}(\epsilon_\theta + \bar{A}\epsilon_\phi) \qquad M_\theta = DA_{11}(\chi_\theta + \bar{A}\chi_\phi) \qquad (10)$$

$$N_{\theta\phi} = BA_{11}(1 - \bar{A})\epsilon_{\theta\phi} \qquad M_{\theta\phi} = DA_{11}(1 - \bar{A})\chi_{\theta\phi} \qquad (11)$$

where B, D are the axial and flexural rigidities in the fully plastic range, given by $B = E_s t/(1 - \nu_p{}^2)$ and $D = E_s t^3/12(1 - \nu_p{}^2)$, with ν_p being the full plastic value of Poisson ratio ($= \frac{1}{2}$) and E_s the secant modulus.

A_{11} is a component of the plasticity coefficient matrix A_{ij}, which, for the spherical case, has the following form[1]):

$$\begin{pmatrix} (3E_t/E_s + 1)/4 & (3E_t/E_s - 1)/2 & 0 \\ (3E_t/E_s - 1)/2 & (3E_t/E_s + 1)/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12)$$

where E_t is the tangent modulus. Finally, $\bar{A}$ in Eqs. (9–11) is a plasticity parameter given by

$$\bar{A} = (\tfrac{1}{2})A_{12}/A_{11} = (3E_t/E_s - 1)/(3E_t/E_s + 1) \qquad (13)$$

It is clear from Eqs. (9–11), with the definitions of the various coefficients, that the preceding relationships can be carried over to the elastic range with the following modifications: $E_t = E_s = E$, the elastic modulus; hence $A_{11} = 1$. B, D now refer to axial and flexural rigidities given by $Et/(1 - \nu_e{}^2)$ and $Et^3/12(1 - \nu_e{}^2)$, respectively (ν_e being the elastic Poisson's ratio). $\bar{A}$ is to be replaced by ν_e, the elastic Poisson ratio.

With foregoing modifications, all the results that are obtained in the plastic case can be written readily for the elastic case. Then the stability problem of the spherical plates reduces to the solving of Eqs. (4) and (7). These can be modified to yield a pair of coupled equations in F and w by making use of the stress-strain relationships [Eqs. (9–11)] and the strain-displacement relationships [Eqs. (1–3, 8)]. The final form of these governing equations is

$$\nabla^2\nabla^2 F = (E_s t/A_{11}R)\nabla^2 w \qquad (14)$$

$$DA_{11}\nabla^2\nabla^2 w + (1/R)\nabla^2 F + p + \sigma t\nabla^2 w = 0 \qquad (15)$$

Before we deal with Eqs. (14) and (15) as such, it is advantageous to study the flat-circular-plate and full-sphere cases first.

Flat Circular Plate

For a flat circular plate under axial compression with $R \to \infty$ in Eqs. (14) and (15), we obtain the following governing equation:

$$\nabla^4 w + (\sigma t/DA_{11})\nabla^2 w + (p/DA_{11}) = 0 \qquad (16)$$

If we let $p = 0$ in Eq. (16), and note that σ is the edge compressive stress, we find that

$$\nabla^2(\nabla^2 + \alpha^2)w = 0 \qquad (17)$$

where $\alpha^2 = (\sigma t/DA_{11})$. The general solution of Eq. (17), with the requirement that w, $(1/r)\partial w/\partial r$ and $\partial^2 w/\partial r^2$ be finite at $r = 0$, can be written as

$$w = \sum_{n=0}^{\infty} [C_{0n}r^n + C_{1n}J_n(\alpha r)] \cos n\theta \qquad (18)$$

when n represents the number of nodal lines on the deformed surface.

For the typical boundary conditions of simple support and full edge fixity at $r = 1$, we obtain the following characteristic equations for determining α. For a simply supported edge ($w = 0$, $M_r = 0$),

Plastic

$$\alpha J_n(\alpha) - (1 - \bar{A})J_{n+1}(\alpha) = 0 \qquad \alpha = (\sigma t/DA_{11})^{1/2} \tag{19a}$$

Elastic

$$\alpha J_n(\alpha) - (1 - \nu_e)J_{n+1}(\alpha) = 0 \qquad \alpha = (\sigma t/D)^{1/2} \tag{19b}$$

For a completely fixed edge ($w = 0$, $\partial w/\partial r = 0$),

Plastic

$$J_{n+1}(\alpha) = 0 \qquad \alpha = (\sigma t/DA_{11})^{1/2} \tag{20a}$$

Elastic

$$J_{n+1}(\alpha) = 0 \qquad \alpha = (\sigma t/D)^{1/2} \tag{20b}$$

For a given n, the roots of Eqs. (19) and (20) correspond to increasing number of nodal circles. Thus the first (lowest root) of Eq. (19) for $n = 0$ would correspond to the axisymmetric case with one nodal circle at the edge. If m denotes the number of nodal circles, then we can characterize the solutions of Eqs. (19) and (20) by their m, n number. Figure 1 shows some typical shapes of the modes taken from Ref. 4.

Table 1 shows some of the typical values, for the first four modes, of an elastic buckling coefficient $k = \sigma t d^2/\pi^2 D = \alpha^2 d^2/\pi^2$, where d is the diameter of the plate. Since all linear dimensions are normalized with respect to the base radius, k becomes equal to $4\alpha^2/\pi^2$. In Eq. (19b), ν_e has been taken equal to 0.3.

In order to separate the plasticity effects, it is useful to define a plasticity reduction factor $\bar{\eta}$, following Ref. 1, given by

$$\bar{\eta} = (\sigma/D)_{\text{plastic}}/(\sigma/D)_{\text{elastic}} \tag{21}$$

In Table 2, we have the results of Eq. (19a) with $n = 0$ given in terms of $\bar{\eta}$ for different E_t/E_s ratios.

Full Sphere

Equations (14) and (15) are transformed, after operating with ∇^2, into the following:

$$\nabla^6 w + (\sigma t/DA_{11})\nabla^4 w + (E_t t/A_{11}{}^2 DR^2)\nabla^2 w = 0 \tag{22}$$

$$\nabla^6 F + (\sigma t/DA_{11})\nabla^4 F + (E_t t/A_{11}{}^2 DR^2)\nabla^2 F = -(E_t t/DA_{11}{}^2 R)p \tag{23}$$

We assume a constant stress state throughout the region under pressure given by $\nabla^2 F = -pR$ so that Eq. (23) is satisfied identically.

Then a suitable form for displacement w would be

$$w = C_n J_n(kr) \cos n\theta \tag{24}$$

By substituting Eq. (24) into Eq. (22) and obtaining a minimum condition for $(\sigma t/DA_{11})$, we find that

$$(\sigma t/DA_{11})_{\min} = 2(E_t t/DA_{11}{}^2 R^2)^{1/2} \tag{25}$$

or

$$\sigma_{\text{cr}} = 3(1 - \nu_e{}^2)^{-1/2}\, \eta(Et/R) \tag{26}$$

where

$$\eta = [(1 - \nu_e{}^2)/(1 - \nu^2)]^{1/2}(E_s/E)(E_t/E_s)^{1/2} \tag{27}$$

or

$$\bar{\eta} = (E_t/E_s)^{1/2}$$

In Eq. (27), ν is the current Poisson ratio appearing in $D = E_s t^3/12(1 - \nu^2)$ as in Ref. 1. It is readily seen that by taking $\eta = 1$ in Eq. (26) we get the elastic critical stress.

It is clear that the expression for σ_{cr} in Eq. (26) is independent of n in the expression for w in Eq. (24). Hence, for both axi- and asymmetric modes, we get the same critical stress for the sphere.

Table 1 Elastic buckling coefficients for flat circular plate

	Governing modes			
Edge conditions	$m = 1$, $n = 0$	$m = 1$, $n = 1$	$m = 1$, $n = 2$	$m = 2$, $n = 0$
Simply supported	1.70	5.32	10.05	11.78
Fully fixed	5.95	11.3	16.5	20.0

Spherical Plate

In the case of a spherical plate, where both curvature and boundary affect the buckling stress, a simple assumption on the stress state cannot be made as in the case of the sphere. Hence, we must consider the general solutions of Eqs. (22) and (23) which can be written conveniently as

$$\nabla^2(\nabla^2 + k_1{}^2)(\nabla^2 + k_2{}^2)w = 0 \tag{28}$$

$$\nabla^2(\nabla^2 + k_1{}^2)(\nabla^2 + k_2{}^2)F = -k_1{}^2 k_2{}^2 Rp \tag{29}$$

where

$$k_1{}^2, k_2{}^2 = \alpha^2/2 \pm [(\alpha^2/2)^2 - \beta^2]^{1/2} \tag{30}$$

and

$$\alpha^2 = \sigma t/DA_{11} \qquad \beta^2 = E_t t/DA_{11}{}^2 R^2 \tag{31}$$

It is seen from Eq. (25) that the case of full sphere corresponds to $\alpha^2/2 = \beta$ or $k_1 = k_2$ in Eq. (30). Hence, for the spherical plate we need consider only $k_1 \neq k_2$.

The solutions of Eqs. (28) and (29) can now be written as

$$w = \sum_{n=0}^{\infty} [C_{0n} r^n + C_{1n} J_n(k_1 r) + C_{2n} J_n(k_2 r)] \cos n\theta \tag{32}$$

$$F = -pR/4r^2 - (E_t t/A_{11}R) \sum_{n=0}^{\infty} [A_{0n} r^n + (C_{1n}/k_1{}^2) J_n(k_1 r) + (C_{2n}/k_2{}^2) J_n(k_2 r)] \cos n\theta \tag{33}$$

where the finiteness of w, $(1/r)\partial w/\partial r$, $\partial^2 w/\partial r^2$, F, $(1/r)\partial F/\partial r$, and $\partial^2 F/\partial r^2$ at $r = 0$ is taken into account.

Although it is possible to treat the problem in its entirety, it is advantageous to consider the axisymmetric case first.

Axisymmetric Case

The solution for the axisymmetric case ($n = 0$) is

$$w = C_{00} + C_{10} J_0(k_1 r) + C_{20} J_0(k_2 r) \tag{34}$$

$$F = pR(r^2/4) - (E_t t/A_{11}R)[A_{00} + (C_{10}/k_1{}^2) J_0(k_1 r) + (C_{20}/k_2{}^2) J_0(k_2 r)] \tag{35}$$

In Eqs. (34) and (35), only three constants C_{00}, C_{10}, and C_{20} are of importance, since any specification on the nonvanishing stress resultants N_θ and N_ϕ involves only the derivatives of F, thus not involving A_{00} at all.

Table 2 Plasticity reduction factors for a simply supported flat circular plate

E_t/E_s	$\bar{\eta}$
1.0	1.0
0.75	0.765
0.50	0.526
0.25	0.278
0	0

The usual boundary conditions of simple support and complete edge fixity, which normally give two conditions on w and its derivatives, are not enough to determine the three constants involved. Thus, extra conditions involving the derivatives of F are required. The additional boundary conditions can be specifications on the membrane stress resultants N_θ, N_ϕ or the tangential displacement u_ϕ or its derivative, which can be written in terms of N_θ, N_ϕ through the use of Eqs. (9) and (10). For each of these groups of boundary conditions, we can set up a system of algebraic equations leading to a characteristic equation whose eigenvalues give us the buckling coefficient corresponding to various modes.

A type of simple support for the edge of a shell which is restrained against normal translation and rotation may be represented by the following boundary conditions at $r = 1$:

$$w = 0 \qquad M_\phi = 0 \qquad N_\phi = -pR/2 \tag{36}$$

The characteristic equation corresponding to this set of boundary conditions is given by

$$[k_1{}^3 J_0(k_1)J_1(k_2) - k_2{}^3 J_0(k_2)J_1(k_1)] + (1 - \bar{A})(k_2{}^2 - k_1{}^2)J_1(k_1)J_1(k_2) = 0 \tag{37}$$

However, by taking a stress boundary condition that is slightly different from Eq. (36), but without changing the restraints against normal translation and rotation, we can obtain a considerable simplification in the governing equation. This is given by

$$w = 0 \qquad M_\phi = 0 \qquad N_\phi = N_\theta = -pR/2 \tag{38}$$

The condition on the membrane stresses in Eq. (38) is seen to imply $N_\phi + N_\theta = -pR$ or $\nabla^2 F = -pR$ at the edge $r = 1$. This results in the vanishing of the constant C_{00} in Eq. (34), which is equivalent to considering a fourth-order equation for w. Such a condition then leads to the following characteristic equation:

$$(k_2{}^2 - k_1{}^2)J_0(k_1)J_0(k_2) + (1 - \bar{A})[k_1 J_0(k_2)J_1(k_1) - k_2 J_0(k_1)J_1(k_2)] = 0 \tag{39}$$

It is of interest to note that the roots of Eqs. (38) and (39) are very close to each other as shown in the lower portion of Fig. 2. Particularly for modes higher than the lowest, the values are indistinguishable. This makes it convenient to use Eq. (38) as a suitable model for simply supported edges with a minimum of mathematical complexity, which is particularly advantageous for the plastic buckling case.

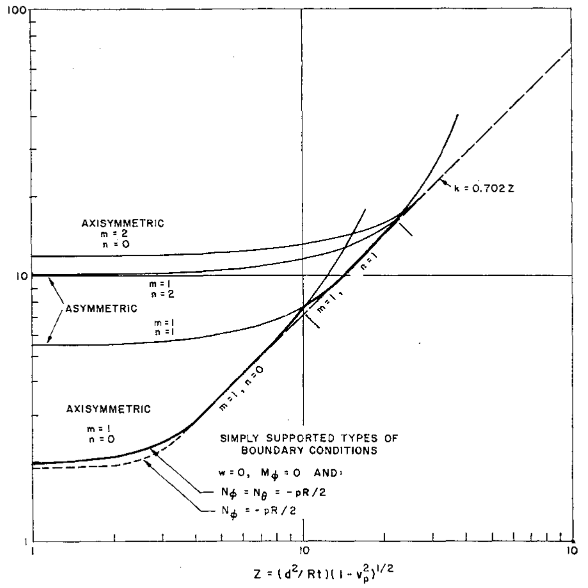

Fig. 2 Elastic buckling coefficient for simply supported types of spherical plates.

Asymmetric Case ($n > 0$)

For $n > 0$, we find now that all of the stress and strain components are no longer independent of θ, the circumferential coordinate, and hence we find that cross terms, like $N_{\theta\phi}$, $\epsilon_{\theta\phi}$, $\chi_{\theta\phi}$, $M_{\theta\phi}$, enter into the problem. Further, the displacement in the circumferential direction u_θ does not vanish in general.

Again, from Eq. (33) it is seen that A_{0n} cannot be ignored if $n > 0$. Thus, with four constants to be determined, we need conditions on other stress resultants like $N_{\theta\phi}$ or the corresponding strains in order to set up the eigenvalue problem. However, if we assume, as in the axisymmetric case, that $\nabla^2 F = -pR$ is a condition that is always valid at the edge, then any requirement that w be zero at the boundary implies the vanishing of C_{0n} for any n. Hence, the eigenvalue problem is simplified. Therefore, we can write the following characteristic equations for the simply supported boundary conditions corresponding to Eq. (38): For the simply supported type ($w = 0$, $M_\phi = 0$, $\nabla^2 F = -pR$ at $r = 1$),

$$(k_2{}^2 - k_1{}^2)J_n(k_1)J_n(k_2) + (1 - \bar{A})[k_1 J_n(k_2)J_{n+1}(k_1) - k_2 J_n(k_1)J_{n+1}(k_2)] = 0 \tag{40}$$

Equation (40) is seen to be the obvious generalization of the corresponding axisymmetric case of Eq. (39). Thus we see, for the boundary conditions described, that Eq. (40) is valid for $n > 0$. Furthermore, from the symmetries of Bessel functions of integral orders, we see that $\pm\beta$ does not affect Eq. (40). Hence, only positive values of k_1, k_2 need be considered.

From Eq. (40), which is the characteristic equation for all values of n, an infinite number of roots can be obtained for a given β. Each nontrivial root for a given n corresponds to an increased number of nodal circles. Thus, the first root for any n would give us a single nodal circle at the edge. If we once again denote the nodal circles by m, we can obtain, from Eq. (40), the eigensolutions (which correspond to the buckling stress) for each value of β (which describes the geometry of the shell suitably) according to their modes (m, n specifications).

Evidently, the solutions corresponding to the elastic case are obtained from Eq. (40) by replacing $\bar{A}$ by ν_e and reinterpreting k_1 and k_2.

Thus, for the elastic case, from Eq. (31) we have

$$\alpha^2 = \sigma t/D \text{ and } \beta^2 = Et/DR^2 \qquad k_1 k_2 = \beta \tag{41}$$

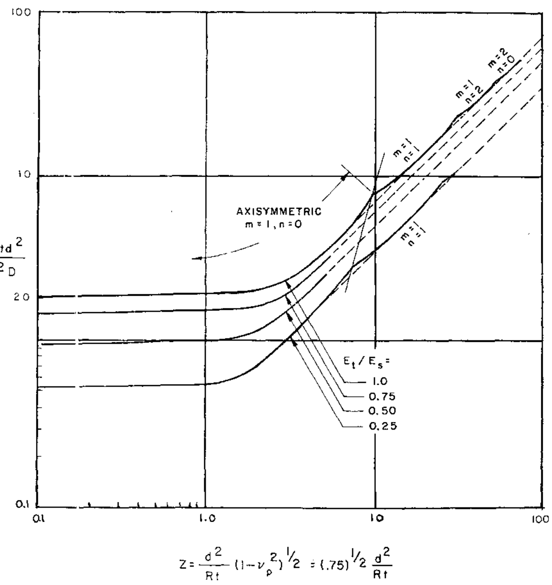

Fig. 3 Plastic buckling coefficient as a function of Z.

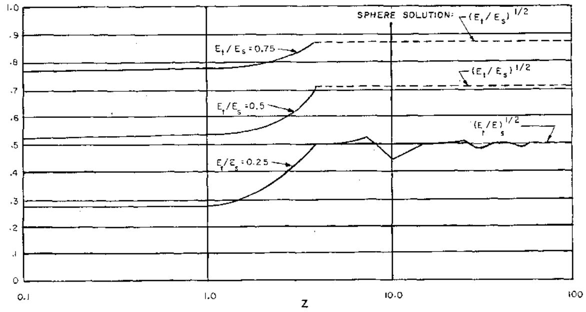

Fig. 4 Plasticity reduction factor as a function of Z.

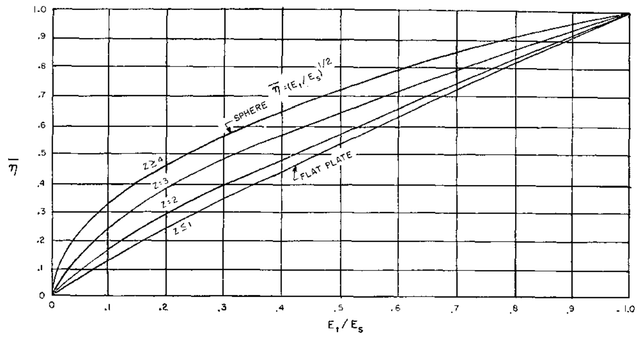

Fig. 5 Plasticity reduction factor as a function of E_t/E_s.

Numerical Results

Equation (40) has been solved for various values of β on a high-speed digital computer for the elastic case with $E_t = E_s = E$, and ν_e, the Poisson ratio, being taken $\nu_e = \nu_p = \frac{1}{2}$ for ease of comparison with other E_t/E_s ratios. Results corresponding to the simply supported type have then been obtained for E_t/E_s ratios of 0.75, 0.50, and 0.25.

For convenience of comparison with the results from cylindrical shell studies,[5] the coordinates have been redefined. A buckling coefficient $k = \sigma t d^2/\pi^2 D$, where d is the base diameter, and a shell parameter Z given by $Z = (d^2/Rt)(1 - \nu_p^2)^{1/2} = (3^{1/2}/2)(d^2/Rt)$, have been used; α and β are related to k and Z in the following manner:

Elastic

$$k = \sigma t d^2/\pi^2 D = d^2\alpha^2/\pi^2 = 4\alpha^2/\pi^2$$

Plastic

$$k = \sigma t d^2/\pi^2 D = (d^2/\pi^2)A_{11}\alpha^2 = 4\alpha^2 A_{11}/\pi^2 \tag{42}$$

Elastic

$$Z = (3^{1/2}/2)d^2/Rt = 2(3)^{1/2}/Rt = (2/3^{1/2})\beta$$

Plastic

$$Z = (3^{1/2})d^2/Rt = 2(3)^{1/2}/Rt = (2/3^{1/2})A_{11}(E_t/E_s)^{-1/2}\beta \tag{43}$$

In Eqs. (42) and (43), since all lengths are normalized with respect to the radius of the base circle, $d = 2$.

For the case of the full sphere, that is, $\alpha^2 = 2\beta$, we have

Elastic

$$k = [4(3)^{1/2}/\pi^2]Z$$

Plastic

$$k = [4(3)^{1/2}/\pi^2](E_t/E_s)^{1/2}Z \tag{44}$$

Figure 2 shows the k-Z plots for simply supported type edges for the elastic case. It is seen that the various modes start from a flat-plate value ($Z = 0$) and, for large values of Z, intersect each other, the intersections being more and more rapid as the higher modes are considered. The minimum curve shown in bold lines in Fig. 2 is seen to be indistinguishable from the straight line for values of Z where these intersections become rapid.

Thus we find that in the transition region between the flat plate and the full sphere, which we signify by the spherical plate solutions, the axisymmetric mode governs; once the value of Z exceeds about ten, the full-sphere solution prevails independent of the buckling mode and also the boundary conditions cease to play any significant role. This behavior is similar to that of cylindrical shells.[5]

Figure 2 shows also the behavior of the stress boundary conditions that are included in the simply supported types of edge restraint. For $m = 1$, $n = 0$, the values lie very close to each other in the transition region, whereas for higher modes, they can hardly be distinguished from each other. Both modes preserve the same pattern of dominant axisymmetric behavior in the transition region.

For the plastic case, which is shown in Fig. 3, the results are studied for the boundary conditions leading to the simplest type of equations, i.e., $w = 0$, $M_\phi = 0$, $\nabla^2 F = -pR$, and the curves are shown therein corresponding to various buckling modes indicated by m, n specifications for E_t/E_s ratios of 1 and 0.25. It is a distinctive feature of the spherical-plate problem that, for sufficiently large valves of Z, there is a single minimum line $k = 0.702\ (E_t/E_s)^{1/2}Z$ corresponding to both the axi- and asymmetric modes. This result is in contrast with the cylindrical problem[5] where there are two minimum lines depending upon whether the governing mode is axi- or asymmetric.

To determine the behavior in the transition region in terms of the plasticity reduction factor defined in Eq. (21), values of $\bar{\eta}$ have been determined for various values of Z using the data presented in Fig. 3. These data are presented in Figs. 4 and 5. For values of Z approximately greater than 4, where the straight line becomes the best approximation for the buckling stress as seen from Fig. 3, $\bar{\eta}$ is equal to $(E_t/E_s)^{1/2}$, the full-sphere solution.

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